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Fractional Integrals of Two Fractional Inverse Trigonometric Functions

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional integral, we find fractional integrals of two fractional inverse trigonometric functions. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of classical calculus results.**

Keywords: **Jumarie type of R-L fractional integral, fractional inverse trigonometric functions, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, mechanics, electrical engineering, viscoelasticity, biology, control theory, dynamics, economics, and other fields [1-16].

The definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [17-20]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on Jumarie type of R-L fractional integral, we find the following fractional integrals of two fractional inverse trigonometric functions:

$$\left({}_0I_x^{\alpha}\right)\left[[arcsin_{\alpha}(x^{\alpha})]^{\otimes_{\alpha} p}\right],$$

and

$$\left({}_0I_x^{\alpha}\right)\left[[arccos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha} p}\right],$$

where $0<\alpha\leq 1,$ and p is a positive integer. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([21])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \quad (2)$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([22])**:** *If* α, β, x_0, C *are real numbers and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \quad (3)$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \quad (4)$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([23])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([24])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \quad (5)$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \quad (6)$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \quad (7)$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \quad (8)$$

Definition 2.5 ([25])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \quad (9)$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \quad (10)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([26])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$.

Definition 2.7 ([27])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function, α-fractional cosine function, and α-fractional sine function are defined as follows:

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}, \tag{13}$$

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

III. MAIN RESULTS

In this section, we find the fractional integrals of two fractional inverse trigonometric functions.

Theorem 3.1: *If* $0 < \alpha \le 1$, *and* p *is a positive integer, then*

$$\left({}_0I_x^\alpha\right)\left[[arcsin_\alpha(x^\alpha)]^{\otimes_\alpha p}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2n+1+p)}\big(arcsin_\alpha(x^\alpha)\big)^{\otimes_\alpha (2n+1+p)} \tag{16}$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}\theta^\alpha = arcsin_\alpha(x^\alpha)$, then $sin_\alpha(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)}x^\alpha$, thus $\left({}_0D_\theta^\alpha\right)[sin_\alpha(\theta^\alpha)] = \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]$, and hence $cos_\alpha(\theta^\alpha) = \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]$. Therefore,

$$\left({}_0I_x^\alpha\right)\left[[arcsin_\alpha(x^\alpha)]^{\otimes_\alpha p}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[[arcsin_\alpha(x^\alpha)]^{\otimes_\alpha p} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$

$$= \left({}_0I_\theta^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right]^{\otimes_\alpha p} \otimes_\alpha cos_\alpha(\theta^\alpha)\right]$$

$$= \left({}_0I_\theta^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right]^{\otimes_\alpha p} \otimes_\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha 2n}\right]$$

$$= \left({}_0I_\theta^\alpha\right)\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha (2n+p)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left({}_0I_\theta^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha (2n+p)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2n+1+p)}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha (2n+1+p)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2n+1+p)}\big(arcsin_\alpha(x^\alpha)\big)^{\otimes_\alpha (2n+1+p)}.$$ q.e.d.

Theorem 3.2: *Suppose that* $0 < \alpha \leq 1$, *and* p *is a positive integer, then*

$$\left({}_{0}I_{x}^{\alpha}\right)\left[[arccos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha} p}\right] = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+2+p)}\left(arccos_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}(2n+2+p)}. \quad (17)$$

Proof: Let $\frac{1}{\Gamma(\alpha+1)}t^{\alpha} = arccos_{\alpha}(x^{\alpha})$, then $cos_{\alpha}(t^{\alpha}) = \frac{1}{\Gamma(\alpha+1)}x^{\alpha}$, thus $\left({}_{0}D_{t}^{\alpha}\right)[cos_{\alpha}(t^{\alpha})] = \left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]$, and hence $-sin_{\alpha}(t^{\alpha}) = \left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]$. Therefore,

$$\left({}_{0}I_{x}^{\alpha}\right)\left[[arccos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha} p}\right]$$

$$= \left({}_{0}I_{x}^{\alpha}\right)\left[[arcsin_{\alpha}(x^{\alpha})]^{\otimes_{\alpha} p}\otimes_{\alpha}\left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\right]$$

$$= \left({}_{0}I_{\theta}^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right]^{\otimes_{\alpha} p}\otimes_{\alpha} - sin_{\alpha}(t^{\alpha})\right]$$

$$= \left({}_{0}I_{\theta}^{\alpha}\right)\left[-\left[\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right]^{\otimes_{\alpha} p}\otimes_{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha}(2n+1)}\right]$$

$$= \left({}_{0}I_{\theta}^{\alpha}\right)\left[-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha}(2n+1+p)}\right]$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left({}_{0}I_{\theta}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha}(2n+1+p)}\right]$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+2+p)}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha}(2n+2+p)}$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+2+p)}\left(arccos_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}(2n+2+p)}.$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional integral, we evaluate fractional integrals of two fractional inverse trigonometric functions. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to study the problems in applied mathematics and fractional differential equations.

REFERENCES

[1] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997.

[2] J. Sabatier, O. P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[3] V. E. Tarasov, Review of Some Promising Fractional Physical Models, International Journal of Modern Physics. vol. 27, no. 9, 2013.

[4] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[5] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[6] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics vol.23, pp. 397-404, 2002.

[7] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[8] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[9] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[10] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[11] M. F. Silva, J. A. T. Machado, and I. S. Jesus, Modelling and simulation of walking robots with 3 dof legs, in Proceedings of the 25th IASTED International Conference on Modelling, Identification and Control (MIC '06), pp. 271-276, Lanzarote, Spain, 2006.

[12] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[13] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[14] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[15] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[16] F. Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, vol. 29, no. 1-4, pp. 315-342, 2002.

[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[18] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[19] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[20] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[21] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021..

[22] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.

[23] C. -H. Yu, Study on some properties of fractional analytic function, International Journal of Mechanical and Industrial Technology, vol. 10, no. 1, pp. 31-35, 2022.

[24] C. -H. Yu, Fractional differential problem of some fractional trigonometric functions, International Journal of Interdisciplinary Research and Innovations, vol. 10, no. 4, pp. 48-53, 2022.

[25] C. -H. Yu, Research on two types of fractional integrals, International Journal of Electrical and Electronics Research, vol. 10, no. 4, pp. 33-37, 2022.

[26] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[27] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.